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Rate of Convergence for Bernstein Polynomials Revisited

Z. DITZIAN*

Department of Mathematics, University of Alberta, Edmonton, Alberta, Canada

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DEDICATED TO THE MEMORY OF GÉZA FREUD

1. INTRODUCTION

The Bernstein polynomials given by

$$B_n(f, x) \equiv \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k(1-x)^{n-k} \equiv \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{k,n}(x) \quad (1.1)$$

and the relation between their rate of convergence and the smoothness of the function they approximate is a topic which has been investigated at great length (e.g., [1, 2, 3, 4, 6, 7]). Berens and Lorentz [1] showed in 1972 that, for $0 \leqslant \alpha \leqslant 2$, $0 < \beta < 2$, $X = x(1-x)$ and $f(x) \in C[0, 1]$,

$$X^{-\alpha/2}\,|B_n(f, x) - f(x)| \leqslant \frac{M(f)}{n^{\beta/2}} \quad (1.2)$$

implies, for $[x-t, x+t] \subset [0, 1]$,

$$X^{-\alpha/2}\,|f(x-t) - 2f(x) + f(x+t)| \equiv X^{-\alpha/2}\,|\Delta_t^2 f(x)| \leqslant M_1(f)\left(\frac{t^2}{X}\right)^{\beta/2} \quad (1.3)$$

and that, for $\alpha = \beta$, (1.3) and (1.2) are equivalent. (This latter fact was also proved by R. DeVore [2].) In [3 and 4] I showed that (1.2) and (1.3) are equivalent for $0 \leqslant \alpha \leqslant \beta < 2$.

Although the most important cases seem to be $\alpha = 0$ (characterizing $\|B_n(f, \cdot) - f(\cdot)\| = O(n^{-\beta/2})$) and $\alpha = \beta$ (characterizing the rate of con-

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vergence for $f \in \text{Lip}^* \beta$, i.e., $\|\Delta_t^2 f(x)\| = O(t^\beta)$), the fact that the question as to what happens when α is bigger than β has remained open was a real annoyance. I was also reminded of that gap by Berens during his visit to Edmonton in the summer of 1983. In this paper it will be shown that (1.2) is equivalent to (1.3) for $0 \leqslant \alpha < 2$ and $0 < \beta \leqslant 2$ (at least in case $\alpha + \beta \leqslant 2$). That is, we drop the condition $\alpha \leqslant \beta$ and the condition $\beta < 2$ as well; the latter is essentially a separate result, being a saturation theorem.

2. Equivalence of an Intermediate Space and Smoothness Condition

The crucial step of the proof that (1.2) and (1.3) are equivalent is characterization of a certain intermediate space. We recall the space C_α as the collection of functions f, $f \in C[0, 1]$ for which $\|(x(1-x))^{-\alpha/2} f(x)\|_{L_\infty[0,1]} \equiv \|f\|_{C_\alpha} < \infty$ and the space C_α^2 a collection of functions in $C[0, 1]$ which are twice differentiable locally in $(0, 1)$ (f and f' are locally absolutely continuous) and for which the seminorm $\|(x(1-x))^{1-(\alpha/2)} f''(x)\|_{L_\infty[0,1]} \equiv \|f\|_{C_\alpha^2}$ is finite. The real (Peetre) interpolation space $(C_\alpha, C_\alpha^2)_\beta$ is the collection of f for which $K(t^2, f)/t^\beta < M(f)$, with norm $\sup_t(K(t^2, f)/t^\beta)$, where $K(t^2, f)$ is given by $K(t^2, f) = \inf_{f=f_1+f_2}(\|f_1\|_{C_\alpha} + t^2 \|f_2\|_{C_\alpha^2})$.

We are now ready to state the main theorem for this paper.

Theorem 2.1. *For* $f(x) \in C[0, 1]$, $X = x(1-x)$, $0 \leqslant \alpha < 2$, $0 < \beta < 2$ *and* $\alpha + \beta \leqslant 2$ *the conditions*

(a) $X^{-\alpha/2} |B_n(f, x) - f(x)| \leqslant Mn^{-\beta/2}$,

(b) $f \in (C_\alpha, C_\alpha^2)_\beta$, *and*

(c) $X^{-\alpha/2} |f(x-t) - 2f(x) + f(x+t)| \leqslant (t^2/X)^{\beta/2}$

are equivalent.

Remark. Theorem 2.1 is our result for $\beta < 2$, for $\beta = 2$ (a) and (c) are equivalent but this is a saturation rather than an inverse theorem and will be proved in the next section.

Proof. The equivalence of (a) and (b) was shown in [1, Theorem 4, p. 703]. Condition (b) implies (c) was shown in [1, Theorem 5, p. 706]. To show that (c) implies (b) it will be sufficient to construct for every τ $f = f_{1,\tau} + f_{2,\tau}$ such that $f_{1,\tau} \in C_\alpha$, $f_{2,\tau} \in C_\alpha^2$, $\|f_{1,\tau}\|_{C_\alpha} \leqslant L\tau^\beta$, and $\|f_{2,\tau}\|_{C_\alpha^2} \leqslant L\tau^{\beta-2}$ and, therefore, $K(\tau^2, f) \leqslant 2L\tau^\beta$. This was not done directly in [4] for $\alpha \leqslant \beta$, where instead, equivalence of (c) with the space $(C_0, C_{2\alpha/\beta}^2)_\beta$ was shown and that space implied (a) which in turn implied (c) and that completed the proof. Here the use of the exact expression of (a), though in a very small part of the proof and though it will not be needed in many of the cases (if $\alpha+\beta<2$), is crucial. (That is, we do not prove (c) implies (b) directly). As in (a), (b), and (c), an additional linear polynomial would not make any difference (we wrote $K(t^2, f)=\inf_{f_1+f_2=f}(\|f_1\|_{C_\alpha}+t^2\|f_2\|_{C^2_\alpha})$ rather than $K(t^2, f)=\inf_{f_1+f_2=f}(\|f_1\|_{C_\alpha}+t^2(\|f_2\|_{C^2_\alpha}+\|f_2\|_C))$ which is equivalent to emphasize that point), we may assume $f(0)=f(1)=0$. We can also concentrate near one of the edges of the interval $[0, 1]$, say 0, as $f(x)=f(x)\,\psi(x)+f(x)(1-\psi(x))$ with $\psi(x)$ a decreasing C^∞ function which satisfies $\psi(x)=1$ for $x\leqslant\frac14$ and $\psi(x)=0$ for $x\geqslant\frac34$ and we will treat here $f(x)\,\psi(x)$ but $f(x)(1-\psi(x))$ can be treated in the same way.

We recall the Stekelov means $f_h(x)\equiv(1/h^2)\int_{-h/2}^{h/2}\int_{-h/2}^{h/2}f(x+u_1+u_2)\,du_1\,du_2$ for which $|f(x)-f_h(x)|\leqslant\max_{t\leqslant h}|\Delta_t^2 f(x)|$ and $f_h''(x)=1/h^2\Delta_h^2 f(x)$. We further define $\psi_k(x)\equiv\psi(4^k x)$ and for $2^{-l-1}<\tau\leqslant 2^{-l}$,

$$g_\tau(x)=\sum_{k=0}^{l-1} f_{2^{-l-k-3}}(x)\,\psi_k(x)(1-\psi_{k+1}(x)).$$

The functions $f_{2^{-l-k-3}}(x)$ are defined for $x\geqslant 2^{-l-k-3}$ but here we need it for $x\geqslant 4^{-k-2}\geqslant 2^{-l-k-3}$ otherwise $1-\psi_{k+1}(x)=0$. We would complete the proof if we showed

$$|X^{-\alpha/2}(f(x)\,\psi(x)-g_\tau(x))|\leqslant L\tau^\beta \tag{2.1}$$

$$|X^{1-(\alpha/2)}g_\tau''(x)|\leqslant L\tau^{\beta-2} \tag{2.2}$$

for all x with L independent of τ. We will show (2.1) only for $x\geqslant 3\cdot 4^{-l-1}$ (and (2.2) for all x).

To prove (2.1) we remember that

$$f(x)\,\psi(x)=\sum_{k=0}^{l-1} f(x)\,\psi_k(x)(1-\psi_{k+1}(x))+f(x)\,\psi_l(x) \tag{2.3}$$

and therefore we have to show for $2^{-l-1}<\tau\leqslant 2^{-l}$,

$$X^{-\alpha/2}\left|\sum_{k=0}^{l-1}(f(x)-f_{2^{-l-k-3}}(x))\,\psi_k(x)(1-\psi_{k+1}(x))\right|\leqslant L\tau^\beta \tag{2.4}$$

for all x and $X^{-\alpha/2}\,|f(x)|\leqslant K_2\tau^\beta$ only for $x\leqslant 3\cdot 4^{-l-1}$, i.e., on the support of $\psi_l(x)$. We will prove (2.4) for all x and therefore (2.1) for $3\cdot 4^{-l-1}<x$. In the sum constituting (2.4) for any given x all but at most two terms are equal to zero as the function $\psi_k(x)(1-\psi_{k+1}(x))$ is different from 0 only for $4^{-k-2}<x<3\cdot 4^{-k-1}$. We will use the inequality $|f(x)-f_h(x)|\leqslant KX^{(\alpha-\beta)/2}h^\beta\leqslant K(\frac43)\,X^{\alpha/2}x^{-\beta/2}h^\beta$ for $x<\frac34$. We have now for $4^{-k-2}\leqslant x\leqslant 3\cdot 4^{-k-1}$,

$$X^{-\alpha/2}\,|f(x)-f_{2^{-l-k-3}}(x)|\leqslant K\tfrac43(2^{-l-k-3})^\beta(4^{-k-2})^{-\beta/2}\leqslant K_1 2^{-l\beta}\leqslant K_2\tau^\beta$$

and L of (2.4) for $x\geqslant 3\cdot 4^{-l-1}$ is $2K_2$. We now prove (2.2) (for all x).

Again only at most two terms being different from 0 in the sum defining g_τ, and therefore for $4^{-k-1} \leqslant x \leqslant 4^{-k}$,

$$X^{1-\alpha/2}\left(\frac{d}{dx}\right)^2 (g_\tau(x)) = X^{1-\alpha/2}\left(\frac{d}{dx}\right)^2 [f_{2^{-l-k-3}}(x) + (f_{2^{-l-k-2}}(x) - f_{2^{-l-k-3}}(x))\,\psi_k(x)]$$

and for $x \leqslant 4^{-l-1}$, $g_\tau(x) = 0$ (and therefore $g_\tau''(x) = 0$). Using $f_h''(x) = h^{-2}\Delta_h^2 f(x)$ and therefore for $x > h$, $|X^{1-\alpha/2} f_h''(x)| \leqslant KX^{1-\alpha/2}h^{-2}h^\beta X^{(\alpha-\beta)/2} = KX^{1-\beta/2}h^{-2+\beta}$, using which we may write for $4^{-k-1} < x < 4^{-k}$,

$$\begin{aligned}\left|X^{1-\alpha/2}\left(\frac{d}{dx}\right)^2 g_\tau(x)\right| &= 3KX^{1-\beta/2}(2^{-l-k-3})^{-2+\beta} \\ &\quad + 2X^{1-\alpha/2}\left|\frac{d}{dx}(f_{2^{-l-k-2}}(x) - f_{2^{-l-k-3}}(x))\right| |\psi_k'(x)| \\ &\quad + X^{1-\alpha/2}\,|f_{2^{-l-k-2}}(x) - f_{2^{-l-k-3}}(x)|\,|\psi_k''(x)| \\ &= I_1 + I_2 + I_3.\end{aligned}$$

We have $I_1 \leqslant K_1(2^{-l})^{\beta-2} \leqslant K_2\tau^{\beta-2}$. Using the estimate of $|f - f_h|$ to get an estimate for $|f_{h_1} - f_{h_2}|$ and the estimate $|\psi_k''(x)| \leqslant A(4^k)^2$ (A is independent of k), we have for $4^{-k-1} \leqslant x \leqslant 4^{-k}$,

$$\begin{aligned}I_3 &\leqslant X^{1-\alpha/2}K\tfrac{4}{3}[(2^{-l-k-3})^\beta(4^{-k-2})^{-\beta/2} + (2^{-l-k-2})^\beta(4^{k-1})^{-\beta/2}]\,A4^{2k} \\ &\leqslant K_3(2^{-l})^{\beta-2} \leqslant K_4\tau^{\beta-2}.\end{aligned}$$

To get the estimate of I_2, we use the estimate for $\phi, \phi'' \in C[a, b]$ given by $\|\phi'\|_{C[a,b]} \leqslant M(b-a)^{-1}\,\|\varphi\|_{C[a,b]} + (b-a)\,\|\phi''\|_{C[a,b]})$, where M does not depend on b and a (see, e.g., [4, p. 310]) with $\phi(x) = f_{2^{-l-k-2}}(x) - f_{2^{-l-k-3}}(x)$, $b = 4^{-k}$, $a = 4^{-k-1}$, and the estimate of ϕ and ϕ'' as in I_1 and I_3. We have $I_2 \leqslant K\tau^{\beta-2}$ when we recall $|\psi_k'(x)| \leqslant A_1 4^k$ in addition to the above. For $4^{-l-1} < x \leqslant 4^{-l}$ we estimate $X^{1-\alpha/2}g_\tau''(x)$ similarly but we have to use $|f(x)| \leqslant M_1 X^{(\alpha+\beta)/2}$ which we will prove in Lemma 2.2. It looks at this stage that we do not have yet "(c) implies (b)," but we do have $|X^{-\alpha/2}(f(x) - g_\tau(x))| \leqslant K\tau^\beta$ for $3\cdot 4^{-l-1} \leqslant x \leqslant 1 - 3\cdot 4^{-l-1}$ or $3\tau^2 \leqslant x \leqslant 1 - 3\tau^2$ and $|X^{1-\alpha/2}g_\tau''(x)| \leqslant K\tau^{\beta-2}$ always. We choose now for a given n, $\tau_n = 1/3\sqrt{n}$ and for $3\tau^2 \leqslant y \leqslant 1 - 3\tau^2$ we have $\frac{1}{3}(1/n) \leqslant y \leqslant 1 - \frac{1}{3}(1/n)$. Therefore,

$$\begin{aligned}|X^{-\alpha/2}(B_n(f, x) - B_{2n}(f, x))| &\leqslant |X^{-\alpha/2}(B_n(f - g_\tau, x) - B_{2n}(f - g_\tau, x))| \\ &\quad + |X^{-\alpha/2}(B_n(g_\tau, x) - B_{2n}(g_\tau, x))| \\ &\equiv I(n, \tau_n) + J(n, \tau_n).\end{aligned}$$

To estimate $I(n, \tau_n)$, we replace $f(y) - g_\tau(y)$ by $\Phi(f, n, y) = (f(y) - g_{\tau_n}(y))\,\chi[1/3n, 1-(1/3n)]$ (where $\chi(A)$ is the characteristic function on A); this would not affect $I(n, \tau_n)$ as $f(0) = f(1) = g_\tau(0) = g_\tau(1) = 0$ and for $1/2n \leqslant y < 1 - (1/2n)$ the expression is the same with Φ as with $f - g_\tau$. The function Φ satisfies $X^{-\alpha/2}\,|\Phi(f, n, x)| \leqslant K\tau^\beta$ for all x.

We have now

$$I(n, \tau_n) \leqslant X^{-\alpha/2}\,|B_n(\Phi(f, n, \cdot), x)| + X^{-\alpha/2}\,|B_{2n}(\Phi(f, n, \cdot), x| \\ = I_1(n, \tau_n) + I_2(n, \tau_n).$$

To estimate $I_i(n, \tau_n)$ we follow [1, p. 704] (it does not matter that $\Phi(f, n, y)$ is not continuous at $1/3n$ and $1-(1/3n)$) to obtain

$$I_1(n, \tau_n) \leqslant X^{-\alpha/2}\,|B_n(\Phi(f, n, \cdot), x| \leqslant X^{-\alpha/2} \sum_{v=0}^{n} \left(\frac{v}{n}\left(1 - \frac{v}{n}\right)\right)^{\alpha/2} P_{v,n}(x)\,\|\Phi\|_{C_\alpha} \\ \leqslant \|\Phi\|_{C_\alpha} \leqslant K\tau_n^\beta$$

or $I(n, \tau_n) \leqslant 2K\tau_n^\beta$. For $J(n, \tau_n)$ the situation is easier as $g_\tau(x)$ is defined on $[0, 1]$ and satisfies $X^{1-\alpha/2}\,|g_\tau''(x)| \leqslant K\tau^{\beta-2}$ and, using the inequality (25) of [1, p. 704], one obtains

$$J(n, \tau_n) \leqslant \left(\frac{7}{n} + \frac{7}{2n}\right) \|X^{1-\alpha/2} g_{\tau_n}''(x)\|_{C[0,1]} \leqslant K_1 \frac{1}{n} \tau_n^{\beta-2}.$$

Recalling $\tau_n = 1/3\sqrt{n}$, $|X^{-\alpha/2}(B_n(f, x) - B_{2n}(f, x))| \leqslant Rn^{-\beta/2}$ (where R does not depend on n or x). Therefore,

$$|X^{-\alpha/2}(B_n(f, x) - f(x))| \leqslant \left| X^{-\alpha/2} \sum_{l=0}^{\infty} (B_{2^l n}(f, x) - B_{2^{l-1}n}(f, x)) \right| \leqslant R_1 n^{-\beta/2}$$

and this is (a) which, according to [1], implies (b) which completes the proof.

We will prove now the estimate of f required in the proof of Theorem 2.1.

LEMMA 2.2. *For $f \in C[0, 1]$, $f(0) = 0$, and $\alpha + \beta < 2$, $X^{-\alpha/2}\,|\Delta_t^2 f(x)| \leqslant M(t^2/X)^{\beta/2}$ implies $|f(x)| \leqslant M_1 X^{(\alpha+\beta)/2}$.*

Proof. Set $t = x$ and write $|f(0) - 2f(x) + f(2x)| \leqslant Mx^{(\alpha+\beta)/2}$. Hence

$$|f(x)| \leqslant M\frac{1}{2} x^{(\alpha+\beta)/2} + \frac{1}{2}\,|f(2x)| \leqslant M\frac{1}{2} x^{(\alpha+\beta)/2} \sum_{l=0}^{L} \frac{2^{(\alpha+\beta)l/2}}{2^l} + \frac{1}{2^L} f(2^L x).$$

Set $\frac{1}{4} \leqslant 2^L x \leqslant \frac{1}{2}$, $2^{-L-2} \leqslant x$ and $|f(2^L x)| \leqslant \|f\|_{C[0,1]}$ and we obtain our result for $\alpha+\beta<2$. For $\alpha+\beta \leqslant 2$ we write $\frac{1}{4}<\zeta<\frac{1}{2}$ and use $|2f(x)-f(2x)| \leqslant Mx$ which implies $|f(\zeta)-2^n f(\zeta/2^n)| \leqslant 2Mx$ or $|f(x)| \leqslant M_1 x$ for $x \leqslant \frac{1}{4}$.

3. The Saturation Case $\beta=2$

The saturation result is given by

THEOREM 3.1. *For* $f \in C[0,1]$ *and* $X=x(1-x)$ *and* $0<\alpha \leqslant 2$ *the conditions*

(a′) $X^{-\alpha/2} |B_n(f,x)-f(x)| \leqslant Mn^{-1}$,

(c′) $X^{1-\alpha/2} |f(x-t)-2f(x)+f(x+t)| \leqslant Kt^2$, *and*

(d) f *and* f' *are locally absolutely continuous in* $(0,1)$ *and* $\|X^{1-\alpha/2} f''(x)\|_{L_\infty[0,1]} \leqslant L$

are equivalent.

Proof. For $\alpha=2$ the theorem was proved by Lorentz in [7, pp. 102–108]. It follows from Berens and Lorentz's paper [1] that $|B_n(f,x)-f(x)| \leqslant (7/n)^{1-\alpha/2}(X/2n)^{\alpha/2} \|f\|_{C^2_\alpha}$ which would prove (d) implies (a′) if $X^{1-\alpha/2}f'' \in C[0,1]$, but there is no difference in the proof if we assume only $X^{1-\alpha/2}f'' \in L_\infty[0,1]$. To show that (c′) implies (d) for $\alpha \neq 2$ is a simple exercise especially as we may use (c′) on the intervals $[2^{-l-1}, 2^l]$ and on $[1-2^{-l}, 1-2^{-l-1}]$ for l integers and recall that on these intervals the weight function $X^{1-\alpha/2}$ is bounded from both sides in the same way in (c′) and (d). To show that (d) implies (c′) we use the Taylor expansion with integral remainder. With no loss of generality, we restrict ourselves to $x \leqslant \frac{1}{2}$ and $0<t<\frac{1}{4}$ and get

$$\begin{aligned} x^{1-\alpha/2} |\Delta_t^2 f(x)| &\leqslant x^{1-\alpha/2} \left[\left| \int_{x-t}^{x} (u-x+t) f''(u)\, du \right| \right. \\ &\quad \left. + \left| \int_x^{x+t} (u-x-t) f''(u)\, du \right| \right] \\ &\leqslant Lx^{1-\alpha/2} \left[\int_{x-t}^{x} \frac{u+t-x}{u^{1-\alpha/2}}\, du + \int_x^{x+t} \frac{x+t-u}{u^{1-\alpha/2}}\, du \right]. \end{aligned}$$

Obviously, $\int_x^{x+t} ((x+t-u)/u^{1-\alpha/2})\, du \leqslant \frac{1}{2}(1/x^{1-\alpha/2})\, t^2$. We also have $\int_{x-t}^{x} ((u-x+t)/u^{1-\alpha/2})\, du \leqslant \frac{1}{2}(1/(x-t)^{1-\alpha/2})\, t^2$ and $\int_{x-t}^{x} ((u-x+t)/u^{1-\alpha/2})\, du \leqslant \int_0^x u^{\alpha/2}\, du = (1/(1+\alpha/2))\, x^{1+\alpha/2}$ which we use for $x>2t$ and $x \leqslant 2t$, respectively. For $x>2t$ we have $x^{1-\alpha/2} |\Delta_t^2 f(x)| \leqslant$

$Lx^{1-\alpha/2}(\frac{1}{2}(1/x^{1-\alpha/2})+\frac{1}{2}(1/(x/2)^{1-\alpha/2}))\,t^2 \leqslant Kt^2$. For $x \leqslant 2t$ we have $x^{1-\alpha/2}\,|\Delta_t^2 f(x)| \leqslant Lx^{1-\alpha/2}((1/(1+\alpha/2))\,x^{1+\alpha/2} + \frac{1}{2}(t^2/x^{1-\alpha/2})) \leqslant L(t^2/2) + L(x^2/(1+\alpha/2)) \leqslant L_1 t^2$.

We are left now with the main part of the proof which is to show (a′) implies (d). Here we utilize a local saturation theorem as the global result [7, p. 102] would not allow us much leeway with a weight function near 0 and 1. We prove it directly as it seems much easier than showing how the proof of an earlier result [6] applies to what we need here.

For $f \in C[0, 1]$ and $f \in C^2[\delta/2, 1-(\delta/2)]$ we have

$$\lim_{n\to\infty} n[B_n(f, x) - f(x)] = \frac{x(1-x)}{2} f''(x) \tag{3.1}$$

uniformly in $[\delta, 1-\delta]$ (see, e.g., [6, Lemma 3.2], but actually the above is a straightforward computation using Taylor's formula for $f(k/n)$ expanded around x). For $X^{-\alpha/2}\,|B_n(f, x) - f(x)| \leqslant Mn^{-1}$ we have $X^{-\alpha/2}\,|B_n(f, x) - f(x)| \leqslant Mn^{-3/4}$ and, therefore, $X^{-\alpha/2}\,|\Delta_t^2 f(x)| \leqslant K(t^2/X)^{3/4}$ or for $x \in [\delta/2, 1-(\delta/2)]$, $|\Delta_t f| \leqslant K_\delta t$. For $g \in C^\infty$ such that $\operatorname{Supp} g \subset [\delta, 1-\delta]$, $|\Delta_t f| \leqslant K_\delta t$ for $x \in [\delta/2, 1-(\delta/2)]$ and $f \in C[0, 1]$, we use the Taylor expansion of $g(x)$ around k/n and recall the actual value of $\int_0^1 (x-(k/n))^i\, P_{n,k}(x)\,dx$ for $i = 0, 1, 2$ [7, p. 106] to obtain

$$\begin{aligned}
&\left|\int_0^1 n(B_n(f, x) - f(x))\, g(x)\,dx\right| \\
&\quad\leqslant \left| n\left\{\sum_{k=0}^{n} f\left(\frac{k}{n}\right) g\left(\frac{k}{n}\right)\frac{1}{n+1} - \int_0^1 f(x)\, g(x)\,dx\right\}\right| \\
&\qquad + \left|\frac{1}{(n+1)(n+2)} \sum_{k=0}^{n} f\left(\frac{k}{n}\right) g'\left(\frac{k}{n}\right)(n-2k)\right| \\
&\qquad + \frac{1}{n(n+1)}\left|\sum f\left(\frac{k}{n}\right) g''(\xi(n, k)\left[\left(\frac{k}{n} - \left(\frac{k}{n}\right)^2\right) + O\left(\frac{1}{n}\right)\right]\right| \\
&\quad\equiv I_1 + I_2 + I_3.
\end{aligned}$$

Now for $\delta/2 < k_0/n < \delta$,

$$\begin{aligned}
I_1 &= n \sum_{k=0}^{n} \left|\int_{k/n+1}^{(k+1)/n+1} \left[f\left(\frac{k}{n}\right) g\left(\frac{k}{n}\right) - f(x)\, g(x)\right] dx\right| \\
&\leqslant n \sum_{k=k_0}^{n-k_0} \left\{\int_{k/n+1}^{(k+1)/n+1} \left|f\left(\frac{k}{n}\right) - f(x)\right| g\left(\frac{k}{n}\right) dx\right. \\
&\qquad \left. + \int_{k/n+1}^{(k+1)/n+1} \left|g(x) - g\left(\frac{k}{n}\right)\right| f(x)\,dx\right\}
\end{aligned}$$

$$\leqslant n\left(K_\delta \frac{1}{n}\|g\|_{C[0,1]} + \frac{1}{n}\|g'\|_{C[0,1]}\|f\|_{C[0,1]}\right) \leqslant K_\delta \|g\| + \|g'\|\,\|f\|,$$

$$I_2 \leqslant \frac{1}{n^2} n \|f\|\,\|g'\|\,n \leqslant \|f\|\,\|g'\|, \quad \text{and} \quad I_3 \leqslant C\frac{1}{n}\|f\|\,\|g''\|.$$

As an alternative to the estimate of $|n\int (B_n(f,x)-f(x))\,g(x)\,dx|$ we could have recalled instead a much more complicated result [6, Lemma 3.4], but an attempt is made here to show that we need only a very crude local result. We have now for $f_l \in C^2$ and for $\langle \psi, \phi\rangle \equiv \int_0^1 \psi(u)\,\phi(u)\,du$,

$$\lim_{n\to\infty} \langle n(B_n(f_l,x)-f_l(x)),\, g(x)\rangle$$
$$= \langle x(1-x)\,f_l''(x), g(x)\rangle = \langle f_l(x), (x(1-x)\,g(x))''\rangle.$$

Obviously, we can choose $f_l \to f$ in the norm $\|f-f_l\|_{C[0,1]} + \text{Sup}_t(1/t)\,\|\Delta_t(f-f_l)\|_{C[\delta/2,1-(\delta/2)]}$ and therefore

$$\lim_{n\to\infty} \langle n(B_n(f,x)-f(x)),\, g(x)\rangle = \lim_{n\to\infty}\lim_{l\to\infty} \langle n(B_n(f_l,x)-f_l(x), g(x)\rangle$$
$$= \lim_{l\to\infty}\lim_{n\to\infty} \langle n(B_n(f_l,x)-f_l(x)),\, g(x)\rangle$$
$$= \lim_{l\to\infty} \langle f_l(x), (x(1-x)\,g(x))''\rangle$$
$$= \langle f(x), (x(1-x)\,g(x))''\rangle.$$

We now recall (a standard technique) that $|X^{-\alpha/2}n(B_n(f,x)-f(x)|$ is bounded in C and therefore $\|n(B_n(f,x)-f(x)\|_{C[\delta,\delta]}$ is bounded and has a weak* accumulation point, say ϕ in $L_\infty[\delta, 1-\delta]$. Consider g as a function in $L_1[\delta, 1-\delta]$; we have $\lim_{n_i\to\infty}\langle n_i(B_n(f,x)-f(x), g(x)\rangle = \langle \phi(x), g(x)\rangle$. (Of course different subsequences may be needed for different g). Now $\langle\phi(x), g(x)\rangle = \langle f, (x(1-x)\,g(x))''\rangle$ for all $g \in C^\infty$ Supp $g \subset [a,b]$ and $[a,b] \subset (0,1)$ and therefore $\phi = x(1-x)\,f''$ in $L_\infty[a,b]$. The crucial point is that the above is true in any subinterval of $[0,1]$ (which does not contain 0 or 1), in particular in $[2^{-l-1}, 2^{-l+1}]$ (and $[1-2^{-l+1}, 1-2^{-l-1}]$), and in that interval $\phi = x(1-x)\,f'' \in L_\infty[2^{-l-1}, 2^{-l+1}]$ the weak* accumulation point of $n(B_n(f,x)-f(x))$ in $[2^{-l-1}, 2^{-l+1}]$ is bounded by $\|n(B_n(f,x)-f(x)\|_{C[2^{-l-1},2^{-l+1}]} \leqslant M(2^{-l-1})^{\alpha/2}$ or $\|x(1-x)\,f''(x)\|_{L_\infty[2^{-l-1},2^{-l+1}]} \leqslant M(2^{-l-1})^{\alpha/2}$ or $\|(x(1-x))^{1-(\alpha/2)} f''(x)\|_{L_\infty[2^{-l-1},2^{-l+1}]} \leqslant 8M$. This being true for all l completes the proof for the estimate on ϕ. The overlapping of the intervals is needed to express that in the intersection we have a unique function ($\phi = x(1-x)\,f''$) and therefore we have one function on $(0,1)$.

Note added in proof. I conjecture that (c) of Theorem 2.1 for $2 < \alpha + \beta$ implies $f(x) = f_1(x) + A_1 x$ and $f(x) = f_2(x) + A_2(1-x)$ where $f_1(x) = 0(x^{(\alpha+\beta)/2})$ as $x \to 0+$, and $f_2(x) = 0((1-x))^{(\alpha+\beta)/2})$ as $x \to 1-$. This will allow us to drop the condition $\alpha + \beta \leqslant 2$ in Theorem 2.1.

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